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Approximate Formulas for Time in Nearly Circular Orbits with Drag

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In this paper we seek approximate closed-form solutions for the flight time in terms of the true anomaly for a satellite in a high near-circular orbit that decays as a result of atmospheric drag. Solutions of this problem are attempted based on three models that approximate the atmospheric density. For the first model and in certain special cases of the other two models, solutions are found that compare favorably with numerical simulations.

I. Introduction

KEPLER'S problem of calculating the flight time of an object in orbit in terms of its true anomaly has been of great interest for hundreds of years. For the circular orbit the problem is trivial; for the general restricted two-body problem the solution in terms of the eccentric anomaly has been known for centuries. The reverse problem of calculating the true anomaly from the time in orbit has also a vast literature.

When atmospheric drag is introduced into the equations of the restricted two-body problem, this problem becomes more difficult. For the special case of near-circular orbits, however, some results obtained by approximating the atmospheric density by mathematical models, and finding certain closed-form solutions of this approximate two-body problem with drag, may be useful in the search for the related closed-form solution for time in terms of true anomaly (Kepler's problem).

Few papers on closed-form solutions of the restricted two-body problem with quadratic drag can be found in the literature. These fall generally into two categories, those that use perturbations or variations of the orbital elements finding solutions in terms of these, or those that simplify the equations of motion for situations in which the orbital motion is mostly tangential. Representative but not exhaustive examples of the former category is the work of Hoots and France [1], King-Helle [2] and more recently Vallado [3]. The second category consists of recent papers by Humi and Carter [4–6] and we shall attempt to build on these in the present work.

We begin with a review of previous results, presenting closed-form expressions for the radial distance of a satellite from the center of attraction based on three approximate atmospheric density models. The section that follows presents new work, revealing the inherent difficulty of this project, and showing solutions in some cases. These solutions are tested against numerical integration, which is displayed graphically and found to be accurate for high altitudes and low drag. The paper ends with some conclusions.

II. Summary of Previous Results

The equations of motion of a satellite in the central force field of another body with quadratic drag are

$$\ddot{\boldsymbol{R}} = -f(R)\boldsymbol{R} - \alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\dot{\boldsymbol{R}} \tag{1}$$

where the satellite is considered as a point mass, and $\boldsymbol{R}$ is the position vector measured from the center of attraction. The upper dots represent differentiation with respect to time t, otherwise the dot represents the scalar product, $R = (\boldsymbol{R}\cdot\boldsymbol{R})^{1/2}$, $f(R) = \mu/R^3$, where μ is the product of the universal gravitational constant and the central mass, α is a constant determined from the drag coefficient, the geometry of the satellite, and the atmospheric density at a specified altitude, and $\rho(R)$ is directly proportional to the atmospheric density at the radial distance R from the center of attraction. It is not difficult to show that the motion of such a point mass is in two dimensions [4] and in polar coordinates (R, θ) the equations of motion become

$$R\ddot{\theta} + 2\dot{R}\,\dot{\theta} = -\,\alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}R\dot{\theta} \tag{2}$$

$$\ddot{R} - R\dot{\theta}^2 = -f(R)R - \alpha\rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\dot{R} \tag{3}$$

If we multiply (2) by R, divide by $R^2\dot{\theta}$, and integrate, we find that

$$R^2\dot{\theta} = J \tag{4}$$

where

$$J = he^{-\alpha\int_0^t \rho(R)(\dot{\boldsymbol{R}}\cdot\dot{\boldsymbol{R}})^{1/2}\,\mathrm{d}t} \tag{5}$$

We designate $\theta_1 = \theta(0)$ as the initial value of θ; h is the value of the instantaneous specific angular momentum J when $t = 0$. We can use (4) to change the independent variable from t to θ. With this change, the chain rule, and some algebra, (2) and (3) produce the following orbit equation:

$$RR''(\theta) - 2R'(\theta)^2 = R^2 - f(R)R^6/J^2 \tag{6}$$

where the prime indicates differentiation with respect to θ, and J is now regarded as a function of θ.

In previous work [4–6] it was shown that these equations can be approximated and simplified by the assumption that the radial component of the velocity $\dot{\boldsymbol{R}}$ is very small compared with its transverse component, as is obviously the case for near-circular orbits. With this simplification closed-form solutions of the orbit equation follow from (6) using the following three models which approximate the atmospheric density when multiplied by an appropriate constant:

$$\rho(R) = 1/R, \qquad \rho(R) = 1/R^2, \qquad \rho(R) = 1/(R-c) \tag{7}$$

In the third model which is the most accurate c is a constant carefully

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chosen to closely approximate the density in an exponential atmosphere (i.e., one in which the density is decreasing exponentially with R; this is a typical local model of the Earth's atmospheric density for R near 7000 km).

We now give a short summary of previous results for each of these cases [4–6]:

A. $\rho(R) = 1/R$

In this case the expression for J is

$$J = he^{-\alpha(\theta-\theta_0)} \tag{8}$$

and the solution of the orbit equation is

$$R = \frac{Pe^{-2\alpha(\theta-\theta_0)}}{[1 + \epsilon e^{-2\alpha(\theta-\theta_0)}\cos(\theta-\theta_0)]} \tag{9}$$

where P, θ_0, and ϵ are constants [4].

Observe that when $\alpha \neq 0$, ϵ is not the orbit eccentricity. In fact even when $\epsilon = 0$ we have

$$R = Pe^{-2\alpha(\theta-\theta_0)} \tag{10}$$

and

$$\frac{dR}{dt} = -2\alpha Pe^{-2\alpha(\theta-\theta_0)}\frac{d\theta}{dt} \tag{11}$$

Because $\dot{\theta} > 0$ this expression can never be zero. It is therefore impossible to start this arc from an initial circular orbit because (11) cannot satisfy the initial condition $\dot{R}(0) = 0$. For very small drag, however, this initial condition can be closely approximated.

We conclude then that a proper interpretation can be assigned only to $\epsilon e^{-2\alpha(\theta-\theta_0)}$ as the "resulting orbit eccentricity" if α is switched to zero at $\theta(t)$. Similarly P is related to $R(\theta_0)$ by the relation $P = R(\theta_0)(1+\epsilon)$.

B. $\rho(R) = 1/R^2$

In this case [5] the expression for J is

$$J(\theta) = he^{-\alpha u} \tag{12}$$

where

$$u(\theta) = \int_{\theta_0}^{\theta}\frac{d\phi}{R(\phi)} \tag{13}$$

The orbit Eq. (6) becomes

$$u''' + u' - \frac{\mu}{h^2}e^{2\alpha u} = 0 \tag{14}$$

Assuming that $|\alpha u| \ll 1$, we can linearize this equation to obtain

$$u''' + u' - \frac{\mu}{h^2}(1 + 2\alpha u) = 0 \tag{15}$$

(We regret that the factor h^2 was inadvertently omitted in these equations in [5]. We restore it here.) From the solution of (15) we obtain

$$R = \frac{Pe^{-a(\theta-\theta_0)}}{1 + \epsilon e^{-\frac{3a(\theta-\theta_0)}{2}}\cos\omega(\theta-\theta_0)} \tag{16}$$

where

$$a = \beta - \frac{1}{3\beta}, \qquad \omega = (4 + 3a^2)^{1/2}/2 \tag{17}$$

and

$$\beta = \left[\frac{[1 + 27(\frac{\alpha\mu}{h^2})^2]^{1/2}}{3\sqrt{3}} + \frac{\alpha\mu}{h^2}\right]^{1/3} \tag{18}$$

In the special case where $\epsilon = 0$ (16) simplifies dramatically:

$$R = Pe^{-a(\theta-\theta_0)} \tag{19}$$

As in the preceding case the initial condition $\dot{R}(0) = 0$ cannot be satisfied but can only be approximated. The interpretation of ϵ and P are similar to those associated with (9).

C. $\rho(R) = 1/(R - c)$

A typical model for the density of the Earth's atmosphere at heights near 7000 km above the center of the Earth is given by

$$\rho_{\text{exp}} = \rho_0 e^{-\frac{(R-R_0)}{H}} \tag{20}$$

where $H = 88.667$ km and $R_0 = 7120$ km. In the following we absorb ρ_0 in α. The curve for the exponential in (20) can be approximated closely by a function of the form $A/(R-c)$ with $A = 115$ km and $c = 7005$ km over a range of 40 km in R. We call this the approximated exponential model. Using this approximation for the density [6] and introducing u as in Eq. (13) we obtain for J the following expression:

$$J = he^{-\alpha[\frac{(2\delta-1)(\theta-\theta_0)}{\delta^2} + \frac{c}{\delta^2}u]} \tag{21}$$

where δ is a number that is chosen so that $|[\delta - (1 - c/R)]/\delta| \ll 1$. In our computations we used $\delta = 0.016, 04$. The orbit Eq. (6) becomes

$$u''' + u' = \frac{\mu}{h^2}\exp\left[\frac{2\alpha(2\delta-1)(\theta-\theta_0)}{\delta^2} + \frac{2\alpha c}{\delta^2}u\right] \tag{22}$$

Approximating the exponential as we did in the previous model we get the following linear equation for u:

$$u''' + u' - \frac{2\alpha\mu c}{h^2\delta^2}u = \frac{\mu}{h^2}\left[1 + \frac{2\alpha(2\delta-1)}{\delta^2}(\theta-\theta_0)\right] \tag{23}$$

Solving this equation the solution of (6) for R can written as

$$R = \frac{Pe^{-a(\theta-\theta_0)}}{1 + \epsilon_1 e^{-\frac{3a}{2}(\theta-\theta_0)}\cos\omega(\theta-\theta_0) + \epsilon_2 e^{-a(\theta-\theta_0)}} \tag{24}$$

where P, θ_0, and ϵ_1 are integration constants and

$$\epsilon_2 = -\frac{P(2\delta-1)}{c} \tag{25}$$

The values of a and ω are given by

$$a = \beta - \frac{1}{3\beta}, \qquad \omega = \frac{\sqrt{3}}{2}\left(\beta + \frac{1}{3\beta}\right) \tag{26}$$

where

$$\beta = \frac{1}{3\delta}\left[\frac{27\mu\delta\alpha c}{h^2} + 3\sqrt{3}\delta\sqrt{\delta^4 + \left(\frac{27\mu\alpha c}{h^2}\right)^2}\right]^{1/3} \tag{27}$$

As to the interpretation of the constants that appear in (24) we note that if the drag coefficient was switched to zero at time t then the resulting orbit eccentricity will be

$$\frac{\epsilon_1 e^{-\frac{3a}{2}[\theta(t)-\theta_0]}}{1 + \epsilon_2 e^{-a[\theta(t)-\theta_0]}}$$

Similarly if no drag is present at all (that is $\alpha = 0$ and hence $a = 0$) then the resulting eccentricity of the orbit will be $\epsilon_1/(1+\epsilon_2)$. P is related to $R(\theta_0)$ by $P = R(\theta_0)(1 + \epsilon_1 + \epsilon_2)$.

III. Time in Orbit

In this section we attempt to derive explicit formulas for the time in orbit as a function of θ for the three atmospheric density models that were discussed in the previous section.

A. $1/R$ Model

To search for a relationship between time and θ we substitute the expressions from (8) and (9) into (4). To simplify the work we shall denote $\theta - \theta_0$ in (9) by ϕ. The entity (4) becomes

$$\dot{\phi} = \frac{h}{P^2} e^{3\alpha\phi}(1 + \epsilon e^{-2\alpha\phi}\cos\phi)^2 \tag{28}$$

Integrating this equation with $\phi(0) = \phi_1 = \theta_1 - \theta_0$ leads to

$$t = \frac{P^2}{h}\int_{\phi_1}^{\phi} \frac{e^{-3\alpha\psi}}{(1 + \epsilon e^{-2\alpha\psi}\cos\psi)^2}\,\mathrm{d}\psi \tag{29}$$

To derive an explicit analytical expression for t we assume that $|\,\epsilon\,| \ll 1$ and use the approximation

$$\frac{1}{(1+x)^2} \approx 1 - 2x$$

for small x. If we let

$$I(\phi) = -\frac{P^2}{h}\left[\frac{e^{-3\alpha\phi}}{3\alpha} + \frac{2\epsilon}{1 + 25\alpha^2} e^{-5\alpha\phi}(\sin\phi - 5\alpha\cos\phi)\right] \tag{30}$$

then an explicit expression for the time in orbit is given by

$$t(\theta) = I(\phi) - I(\phi_1) \tag{31}$$

For the special case in which $\epsilon = 0$ the radial distance R is defined by (10) and the formula for the time in orbit simplifies:

$$t(\theta) = -\frac{P^2}{h}\left(\frac{e^{-3\alpha\phi} - e^{-3\alpha\phi_1}}{3\alpha}\right) \tag{32}$$

This form is preferred over the similar result in [4]. As shown in that reference, if we take the limit as α approaches zero we get the well-known formula for time in circular orbit without drag

$$t(\theta) = \frac{P^2}{h}(\theta - \theta_1) \tag{33}$$

B. $1/R^2$ Model

We consider first the case with $\epsilon = 0$ in (16). Combining (4), (12), (13), and (19) we obtain

$$\dot{\theta} = \frac{h}{P^2} e^{2a(\theta-\theta_0)} \exp\left[-\frac{\alpha}{aP}(e^{a(\theta-\theta_0)} - 1)\right] \tag{34}$$

As before we let $\phi = \theta - \theta_0$ and integrate this equation from ϕ_1 to ϕ. This leads to

$$t = K(\phi) - K(\phi_1) \tag{35}$$

where

$$K(\phi) = -\frac{\alpha^2 e^{-\frac{\alpha}{aP}}}{2a^3 h}\left[\frac{a^2P^2}{\alpha^2}\exp(-2a\phi + \psi) + aP\exp(-a\phi + \psi) + \Gamma(0, -\psi)\right] \tag{36}$$

In this expression $\psi = (\alpha e^{a\phi}/aP)$ and $\Gamma(x, y)$ is the incomplete Gamma function. Furthermore observe that in this formula $a \to 0$ as $\alpha \to 0$ [see (18)] hence it is easy to see that in this limit (35) reduces to (33).

When $\epsilon \neq 0$ the expression for J is given by (12) where

$$u = I(\theta) - I(\theta_0) \tag{37}$$

and

$$I(\theta) = \frac{e^{a(\theta-\theta_0)}}{P}\left\{\frac{1}{a} + \frac{\epsilon e^{-3a/2(\theta-\theta_0)}}{a^2 + 4\omega^2}[4\omega\sin\omega(\theta-\theta_0) - 2a\cos\omega(\theta-\theta_0)]\right\} \tag{38}$$

A corresponding integral expression for the time can be written analytically using some approximations but long and cumbersome expressions are of little practical value. Finding a concise formula as was found in (31) is an open problem.

C. Approximated Exponential Model

For this model we consider only the case $\epsilon_1 = 0$. When $\epsilon_1 \neq 0$ the search for an explicit analytic expression for $t(\theta)$ becomes extremely cumbersome. Although much of the material can be approximated and manipulated by a symbolic algebra package, a useful formula is yet to be found. This, also, is an open problem. The explicit expression for J under the restriction $\epsilon_1 = 0$ is

$$J = h\exp\left[-\frac{\alpha}{\delta^2}(2\delta - 1)(\theta - \theta_0)\right]\exp\left[-\frac{\alpha c}{\delta^2}u\right] \tag{39}$$

where

$$u = \frac{1}{P}\left[\frac{e^{a(\theta-\theta_0)} - 1}{a} + \epsilon_2(\theta - \theta_0)\right] \tag{40}$$

Again introducing $\phi = \theta - \theta_0$, after some manipulations, we obtain from (4)

$$t = \frac{P^2}{h}\int_{\phi_1}^{\phi}\frac{e^{-2a\psi}\exp[B(e^{a\psi} - 1)]\,\mathrm{d}\psi}{(1 + \epsilon_2 e^{-a\psi})^2} \tag{41}$$

where $B = (c\alpha/aP\delta^2)$.

Integrating this expression leads to

$$t = \frac{P^2}{h}[D(\phi) - D(\phi_1)] \tag{42}$$

where

$$D(\phi) = \frac{e^{B\exp(a\phi)}}{a\epsilon_2 e^B(e^{a\phi} + \epsilon_2)} + \frac{(1 + B\epsilon_2)\Gamma[0, -B(e^{a\phi} + \epsilon_2)]}{a\epsilon_2^2 e^{B(1+\epsilon_2)}} - \frac{\Gamma(0, -Be^{a\phi})}{a\epsilon_2^2 e^B} \tag{43}$$

IV. Comparisons with Numerical Integration

Graphical representations for time as a function of θ can be obtained directly by numerical integration of (2) and (3). We present a comparison of these numerical results with those obtained from the analytic formulas developed in the previous section. In this comparison we used the same initial conditions for the numerical integration and the analytic formulas. Unfortunately, explicit (exact) analytic formulas were developed in the previous section only for the cases where $\epsilon = 0$ for the $1/R^2$ model and $\epsilon_1 = 0$ for the approximated exponential model. As a result this comparison can be carried only for a subclass of orbits that satisfy these conditions. It is hoped that further progress will eventually be obtained for orbits with $\epsilon \neq 0$ or $\epsilon_1 \neq 0$ in the appropriate cases. Despite these restrictions, this work could be of value for conditions where α is very small.

Under these conditions the initial value $\dot{R}(0)$ is very near zero, as is the case if the orbits were nearly circular initially. The reader will observe from (11) that it is impossible for $\dot{R}(0)$ to be zero if $\alpha > 0$ although it can be very near zero if α is sufficiently small. If $\epsilon_1 = 0$ in

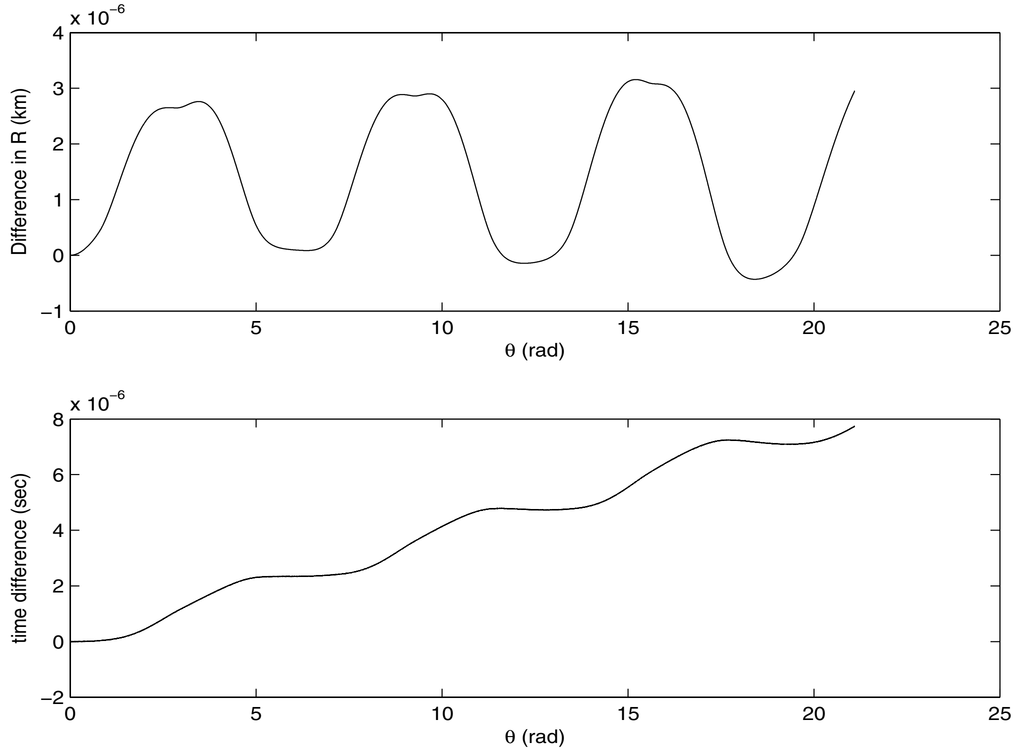

Fig. 1 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). $1/R$ atmospheric model with $\alpha = 10^{-9}$, $\epsilon = 0$.

(24) a similar argument can be made by differentiating this expression.

To provide a uniform reference point for these comparisons the expressions representing the density for the $1/R$ and $1/R^2$ models were normalized, respectively, as follows:

$$\rho = \rho(R_0)\frac{R_0}{R}, \qquad \rho = \rho(R_0)\frac{R_0^2}{R^2} \tag{44}$$

where $R_0 = R(0)$. This normalizes $\rho(R)/\rho(R_0)$ to 1 at R_0. It should be noted that this is the same normalization that is used for the approximated exponential model in (20). We then absorbed $\rho(R_0)$ into α and used the following parameters for all the simulations:

$$R_0 = 7120 \text{ km}, \qquad \alpha = 1 \times 10^{-9}/\text{km} \tag{45}$$

We note that this value of α is too large from a realistic point of view. However, we use it here for illustrative purposes. (Otherwise the orbit decay will be too small to provide a test for our formulas.)

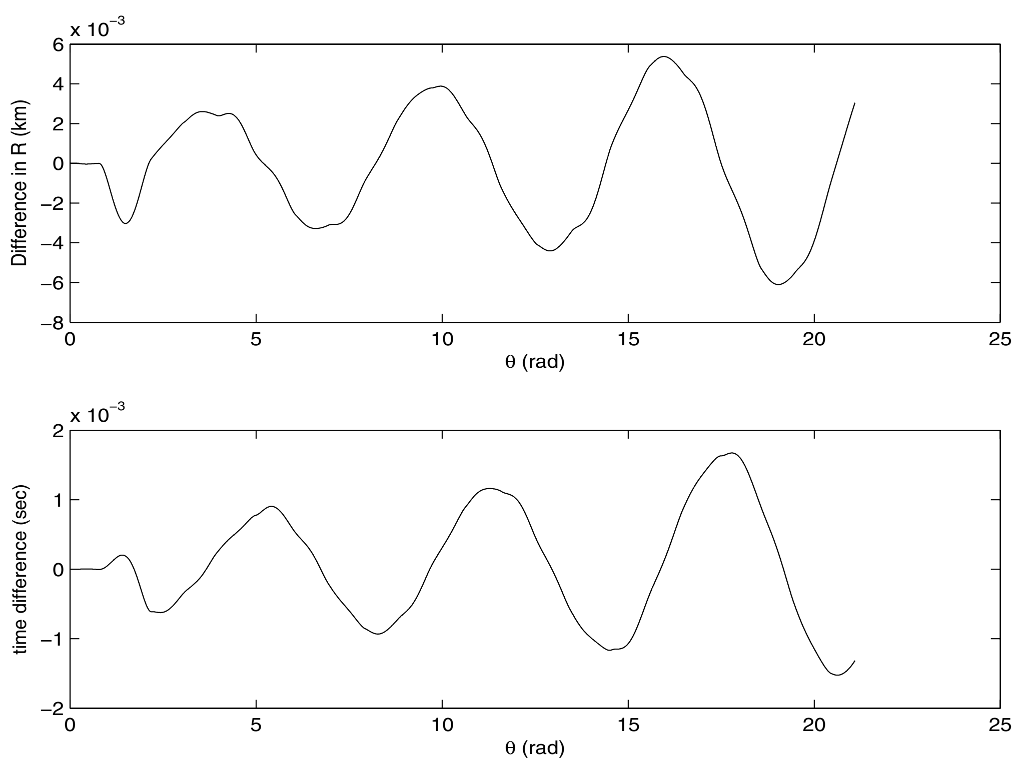

Fig. 2 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). $1/R$ atmospheric model with $\alpha = 10^{-9}$, $\epsilon = 0.0001$.

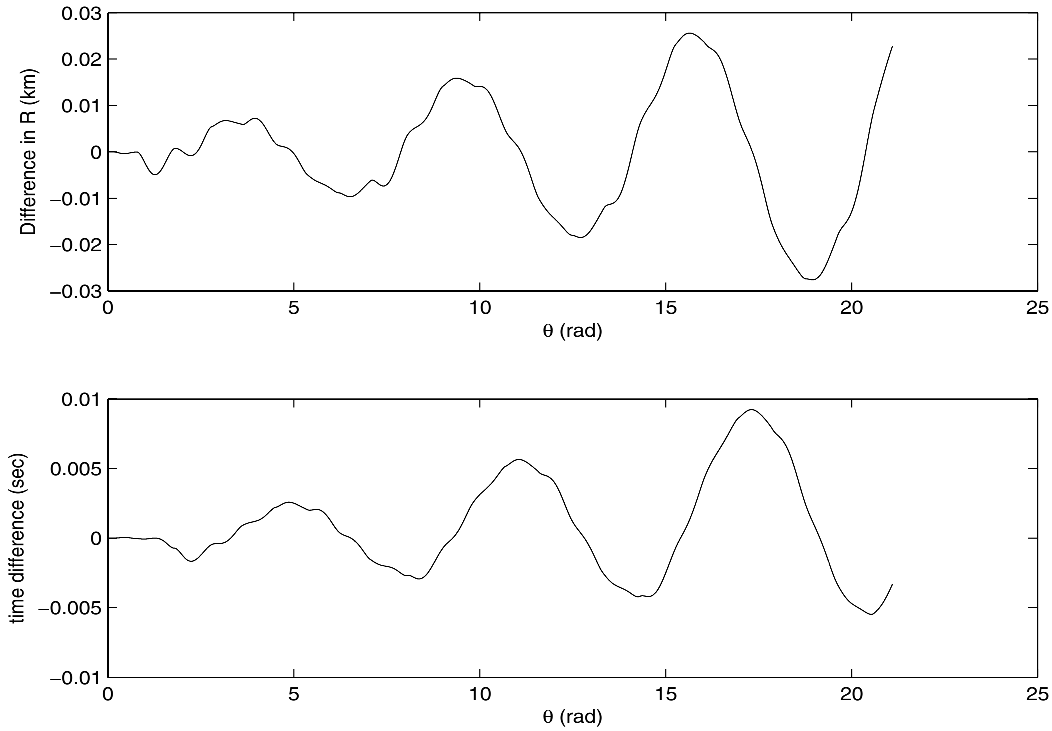

Fig. 3 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). $1/R$ atmospheric model with $\alpha = 10^{-9}$, $\epsilon = 0.001$.

The initial value of $(\mathrm{d}\theta/\mathrm{d}t)$ was chosen so that at time $t = 0$ the gravitational and centrifugal forces balance each other

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(0) = \frac{R_E}{R_0}\sqrt{\frac{g}{R_0}} \tag{46}$$

where $R_E = 6378$ km is the Earth radius and $g = 9.8$ m/s^2 is the acceleration of gravity at sea level.

A. Comparisons with $1/R$ Atmosphere

The differences between the numerical simulations and the analytic formulas for the orbits and the times in orbit are presented for $\epsilon = 0$ and $\theta(0) = 0$ in Fig. 1. The upper part of Fig. 1 shows the difference between R as calculated from (10) and R determined from numerical integration (with the same atmospheric model). The lower part of Fig. 1 shows the difference between the time in orbit from formula (32) and the time in orbit obtained from numerical integration. These figures indicate that the analytical formulas provide an accurate representation. The orbit decay (i.e., the drop in

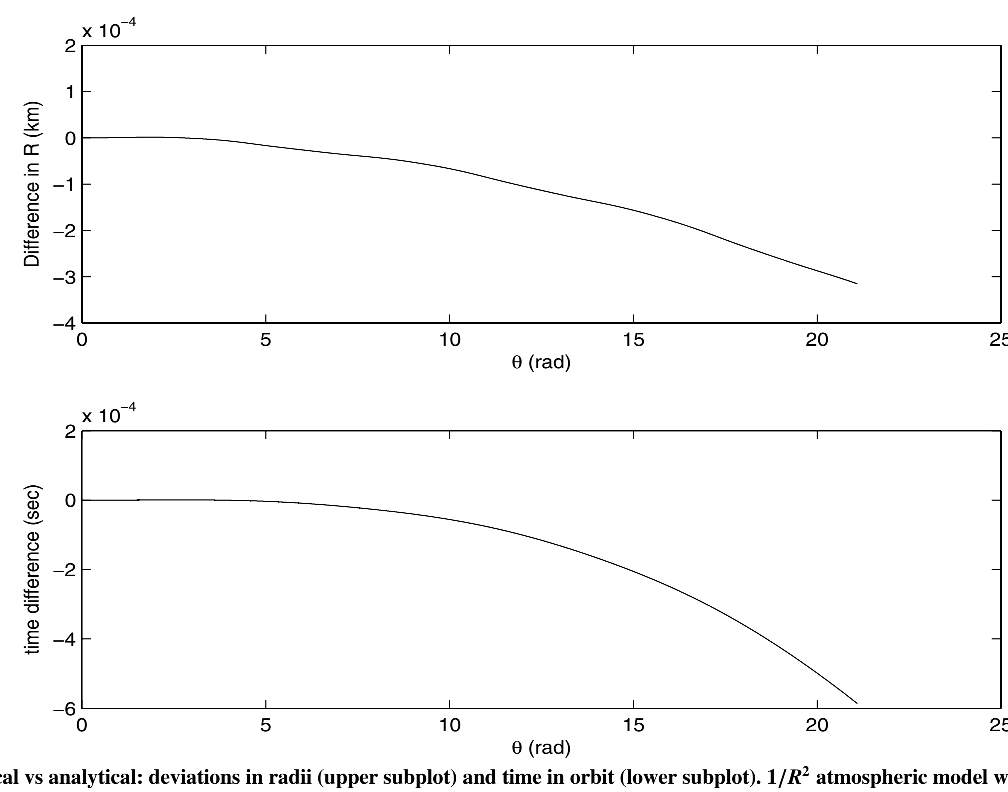

Fig. 4 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). $1/R^2$ atmospheric model with $\alpha = 10^{-9}$, $\epsilon = 0$.

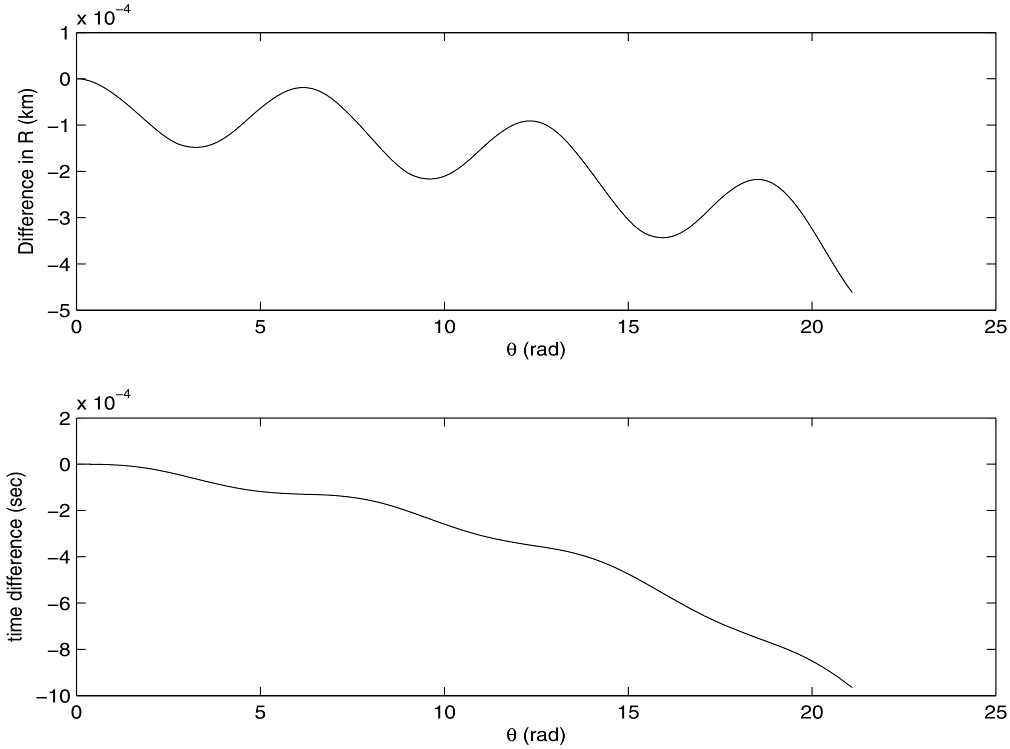

Fig. 5 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). $1/R^2$ atmospheric model with $\alpha = 10^{-9}$, $\epsilon = 0.0001$.

the value of R) over four revolutions was approximately 2 km and the initial radial velocity from (11) was

$$\frac{\mathrm{d}R}{\mathrm{d}t}(0) = -1.07 \times 10^{-4} \ \mathrm{km/s} \tag{47}$$

The period decay per revolution in this simulation is 1.8 s. We see that in this case the differences between the analytic and numerical models are insignificant up to four revolutions.

Similar comparisons are seen in Fig. 2 for $\epsilon = 10^{-4}$ and in Fig. 3 for $\epsilon = 10^{-3}$. In these cases we let $\theta(0) = -\pi/2$. For these comparisons, the analytical formulas are given, respectively, by (9) and (31). The initial radial velocities in these two cases, respectively, were

$$\frac{\mathrm{d}R}{\mathrm{d}t}(0) = -8.58 \times 10^{-4} \ \mathrm{km/s}, \qquad \frac{\mathrm{d}R}{\mathrm{d}t}(0) = -7.62 \times 10^{-3} \ \mathrm{km/s} \tag{48}$$

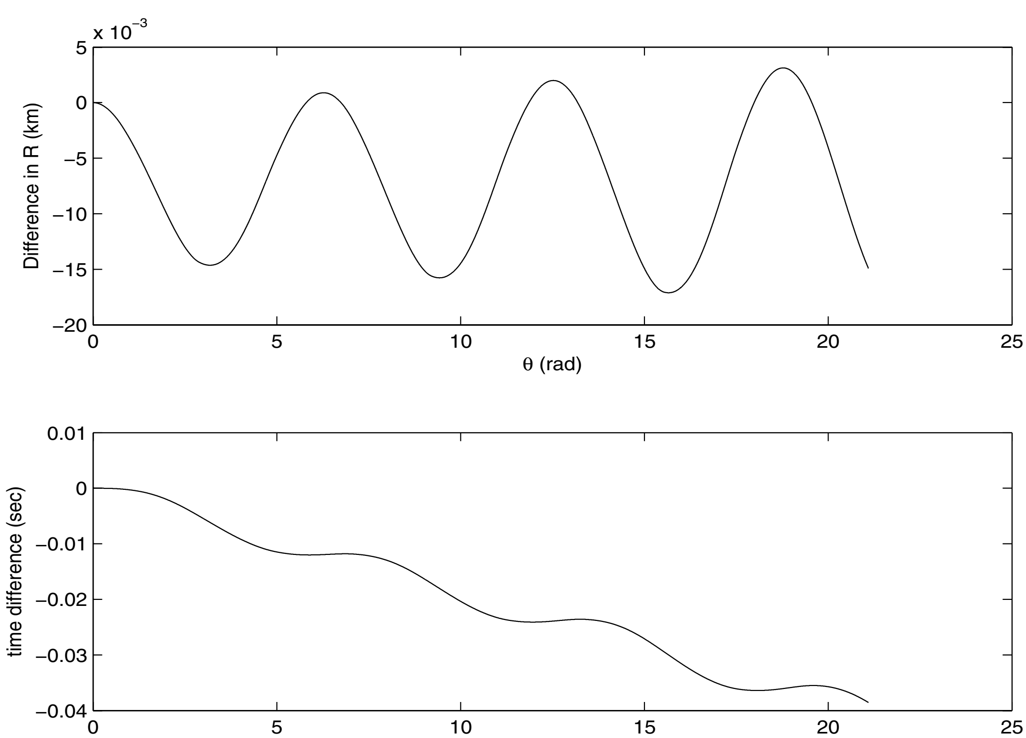

Fig. 6 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). $1/R^2$ atmospheric model with $\alpha = 10^{-9}$, $\epsilon = 0.001$.

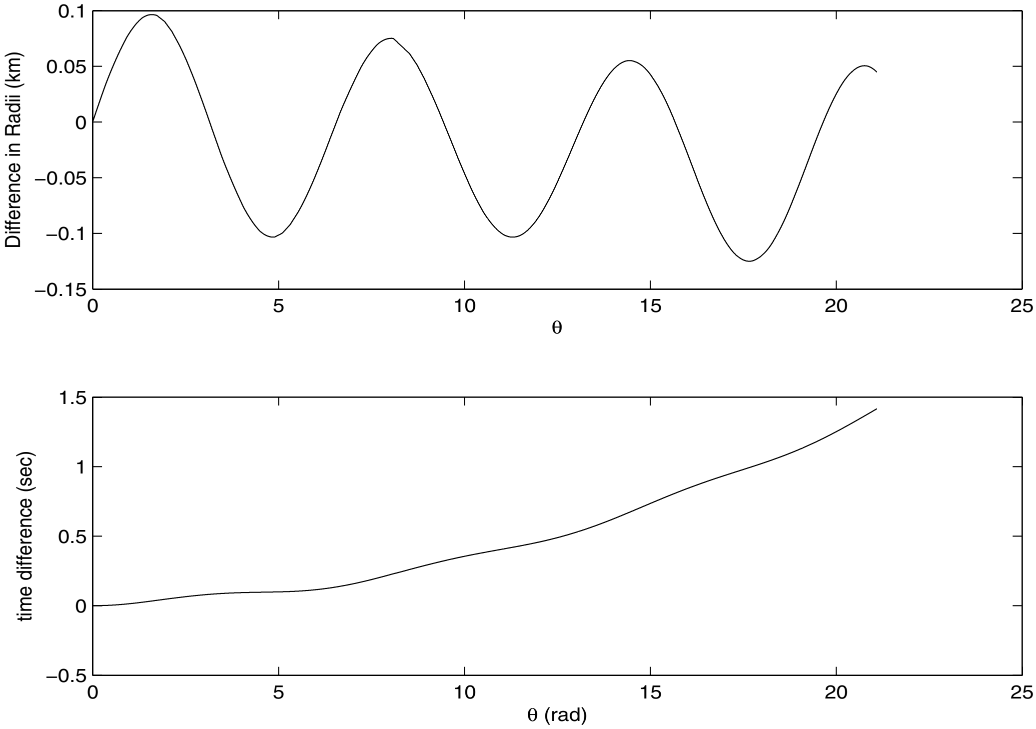

Fig. 7 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). Approximated exponential atmospheric model with $\alpha = 10^{-9}$, $\epsilon_1 = 0$.

We see that as ϵ increases the differences between the analytic and numerical models increase rapidly. This is due to the fact that if $\epsilon \neq 0$ there are arcs for which the approximations that led to our formula (viz. that $\dot{R}$ is much smaller than the transverse velocity) do not hold.

B. Comparisons with $1/R^2$ Atmosphere

We consider first the case where $\epsilon = 0$ and use $\theta_0 = 0$. The initial radial velocity is the same as in (47). The analytical formula for R is (19) and for t it is (35). The orbit decay over four revolutions again

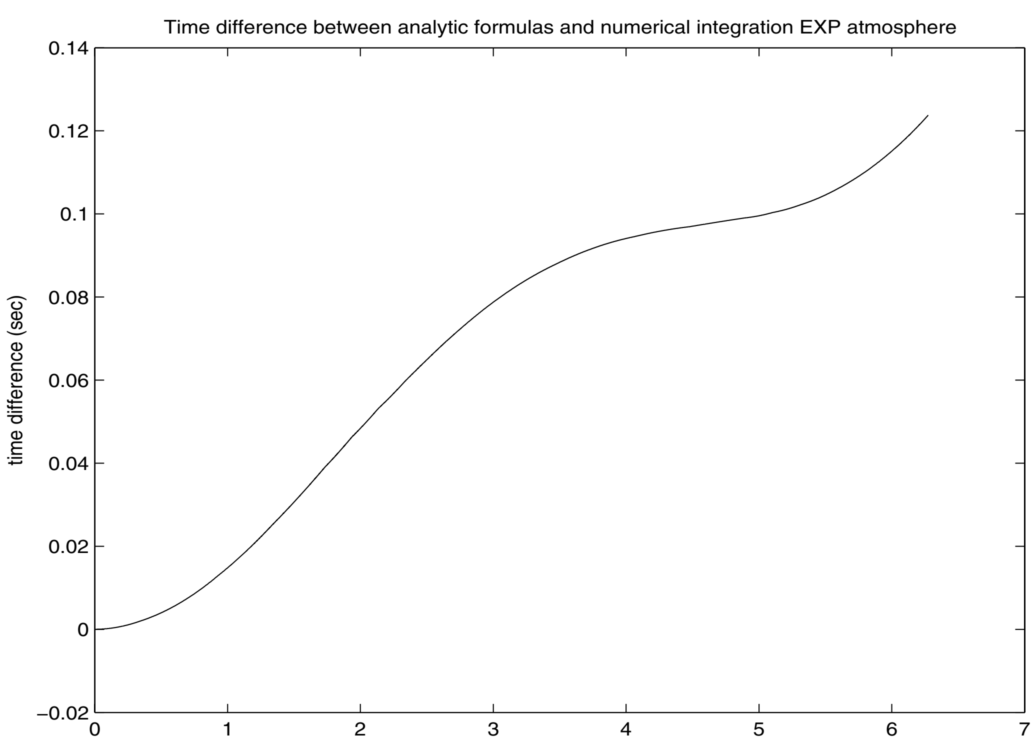

Fig. 8 Time in orbit: difference between analytic formulas and numerical integration over one period. Approximated exponential atmospheric model with $\alpha = 10^{-9}$, $\epsilon_1 = 0$.

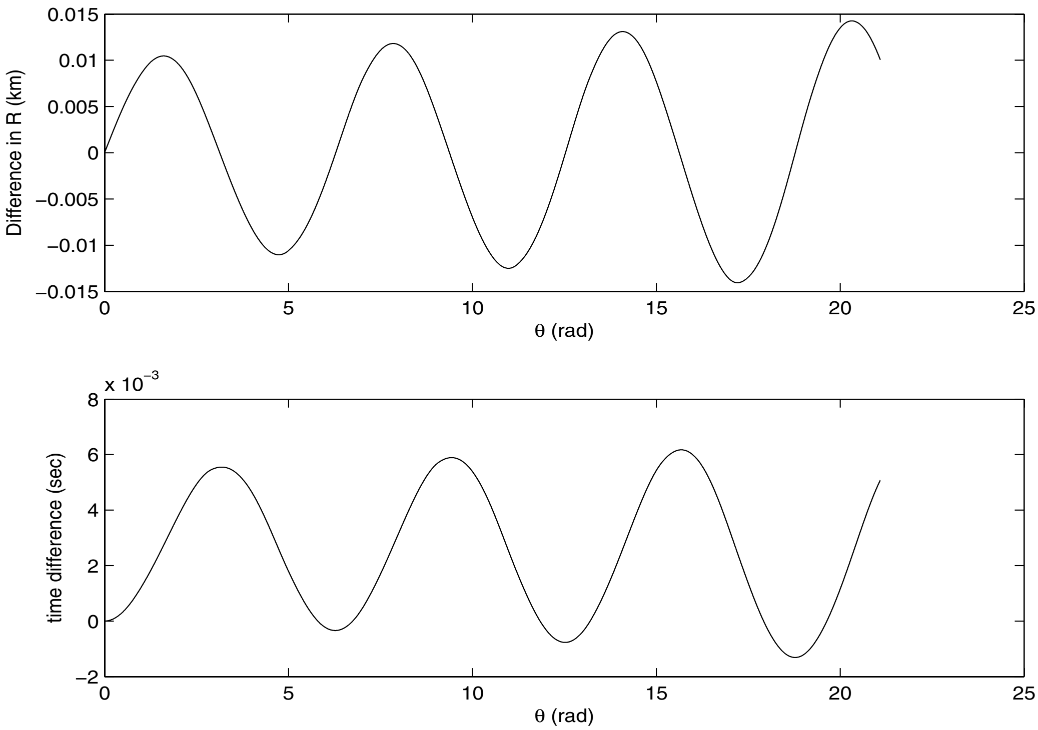

Fig. 9 Numerical vs analytical: deviations in radii (upper subplot) and time in orbit (lower subplot). Exact exponential atmosphere vs $1/R$ model with $\alpha = 10^{-9}$, $\epsilon_1 = 0$.

was approximately 2 km and the period decay per revolution was (approximately) 1.9 s. Respective comparisons with numerical integration (with the same atmospheric model) are seen in Figs. 4. Again the differences are insignificant up to four revolutions. Similar simulations with $\epsilon = 0.0001$, 0.001 are presented in Figs. 5 and 6 respectively.

C. Comparisons with the Approximated Exponential Density

Here we consider differences between the numerical integration of the equations of motion with the atmospheric density given by (20) and the analytic formulas that were derived using the $A/(R - c)$ model for the density. In these comparisons we consider only cases where $\epsilon_1 = 0$ and $\theta_0 = 0$.

The analytical formula for R is (24) with $\epsilon_1 = 0$; for t it is (42). The respective differences in the radii and time in orbit are presented in Fig. 7. Figure 8 shows the differences in the orbital time for the first revolution. In this simulation the orbit decay after three revolution is about 2 km whereas maximum difference in the radii is 0.13 km. The difference between the period of first and second revolutions is (approximately) 2 s. On the other hand, the time in orbit difference after one revolution is 0.124 s. That is the error due to the analytical formula over the first period is 6.1%.

D. Comparisons Between an Exact Exponential Density and the $1/R$ Model

Here we consider a comparison of the numerical integration of the equations of motion with the atmospheric density given by (20) and the analytic formulas that were derived using the $1/R$ model for the density. Although these two models diverge rapidly as R decreases, the difference remains small if the orbit decay is of the order of 1 km over the time period of observations. Using $\alpha = 10^{-10}$ and $\theta_0 = 0$ an orbit decay of 0.2 km transpires over four revolutions. The differences in R and t are presented in Fig. 9. The period decay over one revolution in this simulation is (approximately) 1 s. These figures demonstrate that the analytic formulas that were developed for the $1/R$ model can be useful if the orbital decay is small.

V. Conclusions

We have attempted to derive an analytical expression for the time in orbit of a satellite under the influence of quadratic atmospheric drag using three distinct models that approximate the atmospheric density.

The first model provides the simplest and least accurate representation of the atmospheric density associated with the Earth. For orbits that initially are nearly circular, a solution is found for this problem. Numerical simulations show that this solution is very accurate. The figures demonstrate that for small orbital decay this model can be useful in spite of the fact that it does not provide a realistic representation of the Earth atmospheric density over large variations in R. In fact for a realistic spacecraft the actual drop in R is of the order of 1 m per revolution. Hence this model can used over many periods to approximate this motion.

For the remaining two models the problem becomes very cumbersome analytically. For certain special types of orbits and very small atmospheric drag, however, reasonably simple solutions to this problem are found. Numerical simulations also show these solutions to be very accurate. For greater atmospheric drag or for orbits that are not near circular, the problem of finding reasonably accurate approximate formulas for the time in orbit is very challenging and difficult.

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